PRECALCULUS TOPIC IV
VECTORS IN $\mathbb{R}^2$

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1. Arrows

An *arrow* is a directed line segment in a plane; that is, it is a line segment with one endpoint designated as the *tail* and the other as the *tip*. If the arrow starts at A (A is the tail) and ends at B (B is the tip), we see that such an object is determined by the ordered pair of points (A, B). Denote this arrow by $\widehat{AB}$.

An arrow is determined by three attributes:

- *direction*: this consists of the slope of the line through the two points, together with an orientation along this line;
- *magnitude:* this is the length of the arrow, which is the distance between the tip and the tail;
- *position:* this is described by the placement of the arrow in the plane, and is determined by its tail.

It is possible to define addition of such objects if the tip of one equals the tail of the next; the sum is then defined to be the arrow which starts at the tail of the first and ends at the tip of the second. More precisely,

$$\widehat{AB} + \widehat{BC} = \widehat{AC}.$$

We would like to extend this definition between any two arrows, but in order to do this we need to slide the arrows around via *parallel transport*; that is, we need to disregard the position of the arrow, and only consider its direction and magnitude. We make this precise as follows.

Say that two arrows are *equivalent* if they have the same direction and magnitude, but possibly different positions. Break the set of all arrows in the plane into blocks, where the members of one block consist of all arrows which are equivalent to any other arrow in the block. We call such a block an *equivalence class* of arrows. Any arrow in an equivalence class is called a *representative* of that class.

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2. Vectors

A *vector* is an equivalence class of arrows. The vector represented by an arrow $\widehat{AB}$ is denoted $\overrightarrow{AB}$.

A vector is determined by two attributes:

- direction;
- magnitude.

Thus $\overrightarrow{AB} = \overrightarrow{CD}$ if and only if $\widehat{AB}$ and $\widehat{CD}$ have the same direction and magnitude.

It is now possible to define the addition of any two vectors. Let A, B, C, D be any points on the plane, and consider the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Now $\overrightarrow{CD}$ has a unique representing arrow whose tail is at B, say $\widehat{BE}$. The point E is located geometrically by sliding the arrow $\widehat{CD}$ over so that its tail is at B; or, start at B, and move along the direction of $\widehat{CD}$ for a distance of the magnitude of $\widehat{CD}$, and you will end up at E. Thus $\overrightarrow{CD} = \overrightarrow{BE}$, and it follows our motivation to define

$$\overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{BE}.$$

We put coordinates on the plane. Let A and B are points on the plane, with coordinates given by $A = (a_1, a_2)$ and $B = (b_1, b_2)$ where $a_1, a_2, b_1, b_2 \in \mathbb{R}$.

Now consider two additional points $C = (c_1, c_2)$ and $D = (d_1, d_2)$. A moment's reflection convinces one that $\widehat{AB}$ is equivalent to $\widehat{CD}$ if and only if $d_1 - c_1 = b_1 - a_1$ and $d_2 - c_2 = b_2 - a_2$.

Define $B - A = (b_1 - a_1, b_2 - a_2)$; this is a convenient notation. Let O be the origin, so that $O = (0, 0)$. Let $E = B - A$. Then $\overrightarrow{AB} = \overrightarrow{OE}$. In particular, each vector has a representative which starts at the origin. An arrow whose tail is at the origin is referred to as being in *standard position*. In this way, we can uniquely identify a vector by two real numbers, which are the coordinates of the tip when the tail is at the origin.

Let $\langle x, y\rangle$ denote the vector which is represented by the arrow whose tail is $(0, 0)$ and whose tip is (x, y). The operation of *vector addition* is given by the formula

$$\langle x_1, y_1\rangle + \langle x_2, y_2\rangle = \langle x_1 + x_2, y_1 + y_2\rangle.$$

The *zero vector* is $\langle 0, 0\rangle$; this is the equivalence class of an arrow with no length, that is, of a point. The zero vector serves as an additive identity for vector addition:

$$\langle x, y\rangle + \langle 0, 0\rangle = \langle x + 0, y + 0\rangle = \langle x, y\rangle.$$

Another useful operations on vectors is called *scalar multiplication*. Here we take a representing arrow and stretch it by a factor of t, where $t \in \mathbb{R}$. If t is negative, this reverses the orientation. In coordinates, scalar multiplication is given by

$$t\langle x, y\rangle = \langle tx, ty\rangle.$$

Note that if $t = -1$, scalar multiplication gives the additive inverse of a vector:

$$\langle x, y\rangle + (-1)\langle x, y\rangle = \langle x, y\rangle + \langle -x, -y\rangle = \langle x - x, y - y\rangle = \langle 0, 0\rangle.$$

We now recap and organize what we have discussed.

3. Vector Addition and Scalar Multiplication

Let $\vec{v}_1 = \langle x_1, y_1 \rangle$ and $\vec{v}_2 = \langle x_2, y_2 \rangle$ be vectors in $\mathbb{R}^2$. Define the *vector addition* by

$$\vec{v}_1 + \vec{v}_2 = \langle x_1 + x_2, y_1 + y_2 \rangle.$$

Geometrically, vector addition produces the diagonal of the parallelogram determined by $\vec{v}_1$ and $\vec{v}_2$.

Let $\vec{v} = \langle x, y \rangle$ and $t \in \mathbb{R}$. Define the *scalar multiplication* of $\vec{v}$ by t to by

$$t\vec{v} = \langle tx, ty \rangle.$$

Geometrically, scalar multiplication stretches the vector $\vec{v}$ by a factor of $|t|$, and if t is negative, it reverses its orientation.

Proposition 1. (Properties of Vector Addition)
Let $\vec{v}$, $\vec{w}$, and $\vec{x}$ be vectors in $\mathbb{R}^2$, and let $a, b \in \mathbb{R}$. Let $\vec{0} = \langle 0, 0 \rangle$. Then

- **(a)** $\vec{v} + \vec{w} = \vec{w} + \vec{v}$; *(Commutativity)*
- **(b)** $(\vec{v} + \vec{w}) + \vec{x} = \vec{v} + (\vec{w} + \vec{x})$; *(Associativity)*
- **(c)** $\vec{v} + \vec{0} = \vec{v}$; *(Existence of an Additive Identity)*
- **(d)** $\vec{v} + (-1)\vec{v} = \vec{0}$; *(Existence of Additive Inverses)*

Proposition 2. (Properties of Scalar Multiplication)
Let $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^2$, and let $a, b \in \mathbb{R}$.

- **(a)** $1 \cdot \vec{v} = \vec{v}$; *(Scalar Identity)*
- **(b)** $(ab)\vec{v} = a(b\vec{v})$; *(Scalar Associativity)*
- **(c)** $a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w}$; *(Distributivity of Scalar Mult over Vector Add)*
- **(d)** $(a + b)\vec{v} = a\vec{v} + b\vec{v}$. *(Distributivity of Scalar Mult over Scalar Add)*

Proposition 3. (Additional Properties of Scalar Multiplication)
Let $\vec{v}$ be a vector in $\mathbb{R}^2$, and let $a, b \in \mathbb{R}$. Let $\vec{0} = \langle 0, 0 \rangle$. Then

- **(a)** $0 \cdot \vec{v} = 0$;
- **(b)** $a \cdot \vec{0} = 0$;
- **(c)** $(-a)\vec{v} = -(a\vec{v})$.

We say that two nonzero vectors $\vec{v}$ and $\vec{w}$ are *parallel*, and write $\vec{v} \| \vec{w}$, if arrows representing $\vec{v}$ and $\vec{w}$ lie on parallel line segments. This happens exactly when $\vec{w} = a\vec{v}$ for some nonzero $a \in \mathbb{R}$.

Let $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^2$, and suppose we want the vector which proceeds from the tip of $\vec{v}$ to the tip of $\vec{w}$. Call this vector $\vec{x}$. If we follow $\vec{v}$ and then $\vec{x}$, we arrive at the tip of $\vec{w}$; by the geometric interpretation of vector addition, we see that $\vec{v} + \vec{x} = \vec{w}$. Thus $\vec{x} = \vec{w} - \vec{v}$.

Let $\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$. Then every vector in $\mathbb{R}^2$ can be written as a sum of scalar multiples of these two vectors; in fact,

$$\langle x, y \rangle = x\vec{i} + y\vec{j}.$$

We call $\vec{i}$ and $\vec{j}$ the *standard basis vectors*.

4. Linear Combinations

Let $\vec{v}$ and $\vec{w}$ be vectors. A *linear combination* of $\vec{v}$ and $\vec{w}$ is an expression of the form

$$a\vec{v} + b\vec{w},$$

where a and b are real numbers, and first apply scalar multiplication to each vector, and then vector addition.

Given any two nonzero vectors which are not on the same line, it is possible to write every vector as a linear combination of these two. To see this, let $\vec{v}_1 = \langle x_1, y_1$ and $\vec{v}_2 = \langle x_2, y_2$, and consider a third vector $\vec{v} = \langle x, y\rangle$. We wish to find real numbers a_1 and a_2 such that $a_1\vec{v}_1 + a_2\vec{v}_2 = \vec{v}$. Now two vectors in standard position are equal if and only if their x-coordinates and their y-coordinates are equal, so the last vector equation produces a system of two real equations

$$a_1x_1 + a_2x_2 = x;$$
$$a_1y_1 + a_2y_2 = y;$$

this can be solved if the vectors are not on the same line.

5. Norm of a Vector

The *norm* of a vector is the distance between the tip and the tail of a representing arrow. If the vector is in standard position in $\mathbb{R}^n$, its norm is the distance between the corresponding point and the origin. Thus if $\vec{v} = \langle x, y\rangle$, the norm of $\vec{v}$ is denoted $|\vec{v}|$ and is given by

$$|\vec{v}| = \sqrt{x^2 + y^2}.$$

Synonymous names for this quantity include *modulus*, *magnitude*, *absolute value*, or *length* of the vector.

Example 1. Let $\vec{v} \in \mathbb{R}^2$ be given by $v = \langle 3, 4\rangle$. Find $|\vec{v}|$.

Solution. The length is

$$|\vec{v}| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5.$$

□

A *unit vector* is a vector whose norm is 1. In some sense, a unit vector represents pure direction (without length); if $\vec{u}$ is a unit vector and a is a scalar, then $a\vec{u}$ is a vector in the direction of $\vec{u}$ with norm a.

Let $\vec{v}$ be any nonzero vector. We obtain a unit vector in the direction of $\vec{v}$ simply by dividing by the length of $\vec{v}$. Thus the *unitization* of $\vec{v}$ is

$$\vec{u} = \frac{1}{|\vec{v}|}\vec{v}.$$

Example 2. Let $\vec{v} \in \mathbb{R}^2$ be given by $\vec{v} = \langle 3, 4\rangle$. Find a unit vector in the same direction as v.

Solution. Since $|\vec{v}| = 5$, the unitization of $\vec{v}$ is

$$\frac{\vec{v}}{|\vec{v}|} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle.$$

□

6. Dot Product

Let $\vec{v}_1 = \langle x_1, y_1$ and $\vec{v}_2 = \langle x_1, y_2 \rangle$ be vectors in $\mathbb{R}^2$ We define the *dot product* of $\vec{v}_1$ and $\vec{v}_2$ by the rule

$$\vec{v}_1 \cdot \vec{v}_2 = x_1 x_2 + x_2 y_2.$$

Note that $\vec{v} + \vec{w} \in \mathbb{R}^2$ and $a\vec{v} \in \mathbb{R}^2$, but $\vec{v} \cdot \vec{w} \in \mathbb{R}$.

Proposition 4. (Properties of Dot Product and Norm)
Let $\vec{v}$, $\vec{w}$, and $\vec{x}$ be vectors in $\mathbb{R}^2$, and let $a \in \mathbb{R}$. Let $\vec{0} = \langle 0, 0 \rangle$.

- **(a)** $\vec{v} \cdot \vec{v} = |\vec{v}|^2$;
- **(b)** $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$; *(Commutativity)*
- **(c)** $\vec{v} \cdot (\vec{w} + \vec{x}) = (\vec{v} \cdot \vec{w}) + (\vec{v} \cdot \vec{x})$; *(Distributivity over Vector Addition)*
- **(d)** $a(\vec{v} \cdot \vec{w}) = (a\vec{v}) \cdot \vec{w} = \vec{v} \cdot (a\vec{w})$;
- **(e)** $\vec{v} \cdot \vec{0} = 0$;
- **(f)** $|a\vec{v}| = |a||\vec{v}|$.

Remark. Properties **(a)** through **(f)** are derived directly from the algebraic definitions. Properties **(c)** and **(d)** together are called *linearity of dot product.* □

The geometric interpretation of dot product is as useful as it is unanticipated from the definition. To understand it, we first need to understand the concept of projection.

Given a line L in $\mathbb{R}^2$ and a point P in $\mathbb{R}^2$ not on the line, there is a unique point Q on the line which is closest to the point. The lines L and $\overline{PQ}$ are perpendicular. The point Q is the *projection* of P onto L.

Let $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^2$. There is a unique point on the line through $\vec{w}$ which is the projection of the tip of $\vec{v}$ onto this line. The vector whose tail is the origin and whose tip is this projected point is called the *vector projection* of $\vec{v}$ onto $\vec{w}$. The norm of this vector projection is the distance from the origin to this projected point and is called the *scalar projection* of $\vec{v}$ onto $\vec{w}$. Let $\text{proj}_{\vec{w}}(\vec{v})$ denote the scalar projection of $\vec{v}$ onto $\vec{w}$.

Drop a perpendicular from the tip of $\vec{v}$ onto the line through $\vec{w}$ to obtain a right triangle. If θ is the angle between the vectors $\vec{v}$ and $\vec{w}$, we see that $\text{proj}_{\vec{w}}(\vec{v}) = |\vec{v}| \cos\theta$.

To complete our geometric interpretation of dot product, we need the generalization of the Pythagorean theorem known as the *Law of Cosines.*

Lemma 1 (Law of Cosines)**.** *For a triangle with angles A, B, C and corresponding opposite sides of length a, b, c, we have*

$$c^2 = a^2 + b^2 - 2ab\cos(C).$$

Proof. We show this for the case where B and C are acute angles, the other cases being similar.

Drop a perpendicular from the angle A to the opposite side. Call this distance h. Let m be the distance from C to the perpendicular. Then $a-m$ is the distance from B to the perpendicular. Thus $(a-m)^2 + h^2 = c^2$ and $m^2 + h^2 = b^2$. Substituting $h^2 = b^2 - m^2$ into the first of these yields $a^2 - 2am + b^2 = c^2$. But $m = b\cos(C)$, proving the result. □

Proposition 5. *Let $\vec{v}, \vec{w} \in \mathbb{R}^2$ and let θ be the angle between $\vec{v}$ and $\vec{w}$. Then*

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|\cos\theta.$$

Proof. We prove this result using the Law of Cosines. Consider the triangle whose vertices are the tips of $\vec{v}$ and $\vec{w}$. The vector from $\vec{v}$ to $\vec{w}$ is $\vec{w} - \vec{v}$, so the lengths of the sides of this triangle are $|\vec{v}|$, $|\vec{w}|$, and $|\vec{w} - \vec{v}|$. The Law of Cosines now gives us

$$|\vec{w} - \vec{v}|^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}||\vec{w}|\cos\theta.$$

Since the square of the modulus of a vector is its dot product with itself, we have

$$(\vec{w} - \vec{v}) \cdot (\vec{w} - \vec{v}) = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} - 2|\vec{v}||\vec{w}|\cos\theta.$$

By distributativity of dot product over vector addition and other properties,

$$\vec{w} \cdot \vec{w} - 2\vec{v} \cdot \vec{w} + \vec{v} \cdot \vec{v} = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} - 2|\vec{v}||\vec{w}|\cos\theta.$$

Cancelling like terms on both sides and then dividing by -2 yields

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|\cos\theta.$$

□

Corollary 1. *Let $\vec{v}, \vec{w} \in \mathbb{R}^n$ and let θ be the angle between $\vec{v}$ and $\vec{w}$. Then*

$$\vec{v} \cdot \vec{w} = |\vec{w}|\text{proj}_{\vec{w}}(\vec{v}).$$

If $\vec{u}$ is of unit length, then

$$\vec{v} \cdot \vec{u} = \text{proj}_{\vec{u}}(\vec{v}).$$

Geometrically, the dot product of $\vec{v}$ and $\vec{w}$ is the length of the projection $\vec{v}$ onto $\vec{w}$, times the length of $\vec{w}$.

Example 3. Let $\vec{v} = \langle 5, 2\rangle$ and $\vec{w} = \langle 3, 2\rangle$. Find the scalar and vector projections of $\vec{v}$ onto $\vec{w}$, and find the angle between them.

Solution. We know that $\vec{v} \cdot \vec{w} = |\vec{w}|\text{proj}_{\vec{w}}(\vec{v})$. Thus

$$\text{proj}_{\vec{w}}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|} = \frac{15+4}{\sqrt{9+4}} = \frac{19}{\sqrt{13}}.$$

To obtain $\cos\theta$ from this, divide by $|\vec{v}|$:

$$\cos\theta = \frac{\text{proj}_{\vec{w}}(\vec{v})}{|\vec{v}|} = \frac{19}{\sqrt{13}\sqrt{29}} = \frac{19}{\sqrt{377}},$$

so θ is approximately $12°$. □

We say that $\vec{v}$ is *orthogonal* (or *perpendicular*) to $\vec{w}$, and write $\vec{v} \perp \vec{w}$, if the angle θ between them is a right angle. This happens exactly when the cosine of this angle is zero: $\cos\theta = 0$. Also, by the definition of projection, this happens exactly when the vector projection of $\vec{v}$ onto $\vec{w}$ is the zero vector.

Dot product gives us a test for perpendicularity:

$$\vec{v} \perp \vec{w} \Leftrightarrow \vec{v} \cdot \vec{w} = 0.$$

Note that from this point of view, any vector is perpendicular to the zero vector.

Here is an easy way to find a perpendicular vector.

Proposition 6. $\langle x, y\rangle \perp \langle y, -x\rangle$.

Proof. Compute $\langle x, y\rangle \cdot \langle y, -x\rangle = xy - yx = 0$. □

Proposition 7. *Let* $\vec{v} = \langle a, b\rangle$ *and* $\vec{w} = \langle c, d\rangle$ *be vectors in* $\mathbb{R}^2$*. The area of the parallelogram determined by* $\vec{v}$ *and* $\vec{w}$ *is* $ad - bc$*.*

Proof. The area of a parallelogram of height h and base s is $A = hs$. Consider $\vec{w}$ to be the base; then $s = |\vec{w}|$. Now the height is the scalar projection of $\vec{v}$ onto a vector perpendicular to $\vec{w}$. Let $\vec{x} = \langle -d, c\rangle$; then $\vec{w} \cdot \vec{x} = 0$, so $\vec{w} \perp \vec{x}$. Moreover, $|\vec{w}| = |\vec{x}|$. We have

$$A = hs = |\vec{w}|\frac{\vec{x} \cdot \vec{v}}{|\vec{x}|} = \vec{x} \cdot \vec{v} = ad - bc.$$

□

Example 4. Find the area of the triangle in $\mathbb{R}^2$ with vertices $A = (2, 5)$, $B = (-7, 3)$, and $C = (-2, -4)$.

Proof. The area of a triangle is half of the area of a parallelogram, so if we can translate this into a problem involving vectors, we will be able to use the formula above.

Translate the entire triangle so that one of the points, say C, moves to the origin. We do this by subtracting the vector $\langle -2, -4\rangle$ from the other two points.

Let $\vec{v} = A - C = \langle 4, 9\rangle$ and $\vec{w} = B - C = \langle -5, 7\rangle$. The triangle determined by $\vec{v}$ and $\vec{w}$ is congruent to the original one, and by the previous proposition, its area is $\frac{1}{2}(4(7) - 9(-5)) = \frac{73}{2}$. □

Let L be a line in $\mathbb{R}^2$. A *direction vector* for that line is a vector in the same direction as the line, that is, a vector represented by an arrow such that the tail and the tip of the arrow are on the line. A *normal vector* for a line is a vector which is perpendicular to the direction vector.

Example 5. Let $L : y = 4x - 3$ and $Q : (-2, 5)$. Find the distance from L to Q.

Solution. We wish to find the distance from a point to a line; this, of course, means the shortest distance. Our method is as follows:

(1) Find a point P on the line.
(2) Find a direction vector for the line.
(3) Find a normal vector $\vec{w}$ for the line.
(4) Let $\vec{v} = Q - P$.
(5) Compute $\text{proj}_{\vec{w}}(\vec{v})$.

□

7. Polar Coordinates

Let $\vec{v}$ be a nonzero vector in $\mathbb{R}^2$, represented by an arrow in standard position. The tip of the arrow is a point P in the plane. Let x be the distance from P to the y-axis, and let y be the distance from P to the x-axis; drawing this produces a rectangle. Thus $\vec{v}\langle x, y\rangle$, and we call x and y the *rectangular coordinates* of $\vec{v}$.

We now describe the position of P by a different pair of real numbers.

Let r be the distance from P to the origin. Then

$$r = |\vec{v}| = \sqrt{x^2 + y^2}.$$

Let θ be the angle that the vector $\vec{v}$ makes with the positive x-axis. If $x > 0$, then $\langle x, y\rangle$ lies in the right half plane, and the angle is given by the arctangent of $\frac{y}{x}$. Otherwise, we make adjustments to obtain a unique real number which we call the angle of $\vec{v}$, as follows.

$$\theta = \angle\vec{v} = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0; \\ \arctan 180^\circ - \arctan\left(-\frac{y}{x}\right) & \text{if } x < 0 \text{ and } y > 0; \\ \arctan\left(\frac{y}{x}\right) - 180^\circ & \text{if } x < 0 \text{ and } y < 0; \\ 90^\circ & \text{if } x = 0 \text{ and } y > 0; \\ -90^\circ & \text{if } x = 0 \text{ and } y < 0; \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

The numbers r and θ are called the *polar coordinates* of $\vec{v}$ or P. If $r \in (0, \infty)$ and $\theta \in (-180^\circ, 180^\circ]$, this gives a unique representation of every nonzero point.

We can convert from rectangular to polar coordinates by the definitions above. Moreover, using trigonometry, we see that

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

Polar coordinates simplify certain equations. For example, the equation of the unit circle is $r = 1$, and the equation of the diagonal ray in the first quadrant is $\theta = 45^\circ$.

Example 6. Write the equation $(x-5)^2 + y^2 = 25$ in polar coordinates.

Solution. This equation is equivalent to $x^2 - 10x + 25 + y^2 = 25$, so $x^2 - 10x + y^2 = 0$. Since $x = r\cos\theta$ and $y = r\sin\theta$, the equation becomes

$$r^2\cos^2\theta - 10\cos\theta + r^2\sin^2\theta = 0,$$

which simplifies to

$$r^2 = 10\cos\theta.$$

□

8. Exercises

Exercise 1. Draw the directed line segment $\vec{AB}$. Find and draw the equivalent the vector $\vec{v}$ whose tail is at the origin.

(a) $A(3,1)$, $B(3,3)$
(b) $A(-3,5)$, $B(-2,0)$
(c) $A(0,2)$, $B(5,2)$

Exercise 2. Find the vector sum $\vec{v}+\vec{w}$ and illustrate geometrically.

(a) $\vec{v}=\langle 0,1\rangle$, $\vec{w}=\langle 1,0\rangle$
(b) $\vec{v}=\langle 2,4\rangle$, $\vec{w}=\langle 5,1\rangle$
(c) $\vec{v}=\langle -2,3\rangle$, $\vec{w}=\langle 3,-2\rangle$

Exercise 3. Find $|\vec{v}|$, $\vec{v}+\vec{w}$, $\vec{v}-\vec{w}$, $2\vec{v}$, and $3\vec{v}-2\vec{w}$.

(a) $\vec{v}=\langle 1,2\rangle$, $\vec{w}=\langle 3,4\rangle$
(b) $\vec{v}=\langle -1,-2\rangle$, $\vec{w}=\langle 2,1\rangle$
(c) $\vec{v}=\langle 3,2\rangle$, $\vec{w}=\langle 0,6\rangle$
(d) $\vec{v}=\vec{i}-\vec{j}$, $\vec{w}=\vec{i}+\vec{j}$

Exercise 4. Find a unit vector which has the same direction as $\vec{v}$.

(a) $\vec{v}=\langle 3,4\rangle$
(b) $\vec{v}=\langle 5,-5\rangle$
(d) $\vec{v}=\vec{i}+\vec{j}$

Exercise 5. Express $\vec{i}$ and $\vec{j}$ in terms of $\vec{v}$ and $\vec{w}$.

(a) $\vec{v}=\vec{i}+\vec{j}$, $\vec{w}=\vec{i}-\vec{j}$
(b) $\vec{v}=2\vec{i}+3\vec{j}$, $\vec{w}=\vec{i}-\vec{j}$

Exercise 6. Find $\vec{v}\cdot\vec{w}$.

(a) $\vec{v}=\langle 2,4\rangle$, $\vec{w}=\langle -1,4\rangle$
(b) $\vec{v}=\langle 5,-1\rangle$, $\vec{w}=\langle 7,7\rangle$
(c) $\vec{v}=\langle 7,3\rangle$, $\vec{w}=\langle 3,-7\rangle$
(d) $\vec{v}=\langle -4,1\rangle$, $\vec{w}=\langle 3,6\rangle$

Exercise 7. Find the scalar and vector projections of $\vec{v}$ onto $\vec{w}$.

(a) $\vec{v}=\langle 2,4\rangle$, $\vec{w}=\langle -1,4\rangle$
(b) $\vec{v}=\langle 5,-1\rangle$, $\vec{w}=\langle 7,7\rangle$
(c) $\vec{v}=\langle 2,3\rangle$, $\vec{w}=\langle 2,1\rangle$
(d) $\vec{v}=\langle -4,1\rangle$, $\vec{w}=\langle 3,6\rangle$

Exercise 8. Find the values for $x\in\mathbb{R}$ such that $\vec{v}\perp\vec{w}$.

(a) $\vec{v}=\langle 3,x\rangle$, $\vec{w}=\langle -4,3\rangle$
(b) $\vec{v}=\langle x,12\rangle$, $\vec{w}=\langle x^3,18\rangle$
(c) $\vec{v}=\langle 3,2t\rangle$, $\vec{w}=\langle x,-2\rangle$

Exercise 9. Find the values for $x\in\mathbb{R}$ such that the angle between $\vec{v}=\langle 1,1\rangle$ and $\vec{w}=\langle x,1\rangle$ is $60°$.

Exercise 10. Draw, and find the area of, each triangle with vertices A, B, and C.

(a) $A(0,0)$, $B(5,0)$, $C(0,12)$
(b) $A(0,0)$, $B(2,7)$, $C(-3,5)$
(c) $A(2,3)$, $B(10,7)$, $C(5,12)$

Exercise 11. Consider the points $A(-3,-4)$, $B(-7,6)$, and $C(x,2)$.

(a) Find x so that the area of $\triangle ABC$ is 72.
(b) Find x so that $\angle BAC$ is a right angle.

Exercise 12. Let $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^2$. Give a geometric interpretation of and prove the following formulae:

(a) Cauchy Schwartz Inequality:

$$|\vec{v} \cdot \vec{w}| \leq |\vec{v}||\vec{w}|$$

(b) Triangle Inequality:

$$|\vec{v} + \vec{w}| \leq |\vec{v}| + |\vec{w}|$$

(c) Parallelogram Law:

$$|\vec{v} + \vec{w}|^2 + |\vec{v} - \vec{w}|^2 = 2|\vec{v}|^2 + 2|\vec{w}|^2$$

(Hint for (b) and (c): Use the Cauchy Schwartz Inequality, the distributivity of dot over sum, and the fact that $|\vec{v} + \vec{w}|^2 = (\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w})$.)

Exercise 13. Convert to polar coordinates.

(a) $x = \sqrt{3}$, $y = 1$
(b) $x = -4\sqrt{2}$, $y = 4\sqrt{2}$
(c) $x = \sqrt{6} + \sqrt{2}$, $y = \sqrt{6} - \sqrt{2}$

Exercise 14. Convert to rectangular coordinates.

(a) $r = 1$, $\theta = 30^\circ$
(b) $r = 2$, $\theta = 165^\circ$
(c) $r = 4$, $\theta = -72^\circ$

Exercise 15. Write the equation $x^2 + (y-3)^3 = 9$ in polar coordinates.

Exercise 16. Write the equation $r = \sin\theta$ in rectangular coordinates.

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